Chapter 2. The Mathematics of Power

Weighted Voting

Introduction to Contemporary Mathematics

Math 112

2.1. An Introduction to Weighted Voting

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Definition

A **weighted voting system** is a voting system in which voters are not necessarily equal in terms of the number of votes they control.

We will only consider *yes-no* votes called **motions**.

Elements of a weighted voting system

- **Players:** Voters will be referred to as players. We denote the number of players in a voting system with N and we denote the players by

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- **Weights:** Each player will control a number of votes. That number of votes is called the **weight** of the player. We denote the weights of $P_1, P_2, \ldots, P_N$ to be

$$w_1, w_2, \ldots, w_N \text{ respectively.}$$

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$$w_1, w_2, \ldots, w_N \text{ respectively.}$$
Denote the total number of votes by $V = w_1 + w_2 + \cdots + w_N$.
- **Quota:** The minimum number of votes required to pass a motion. Denote the quota by q.

Notation

Consider a weighted voting system with players $P_1, P_2, \ldots, P_N$ with weights $w_1, w_2, \ldots, w_N$ respectively where

$$w_1 \geq w_2 \geq \cdots \geq w_N.$$

Let q be the quota for the system.

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We denote the weighted voting system as

$$[q : w_1, w_2, \ldots, w_N].$$

Example

Three stockholders in a small company form a Board of Directors to oversee the company. John (P_1) is the largest stock holder with 5 stocks, Ginny (P_2) has 3 stocks, and Ann (P_3) has 2 stocks. They all agree that each stock is worth 1 vote. Thus John has 5 votes, Ginny has 3 votes, and Ann has 2 votes. Suppose that their quota for a motion to pass is 7 votes. The weighted system is then

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$$[7 : 5, 3, 2].$$

Question

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- What happens in the previous example if the quota q is changed from 7 to 11?

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Definition (Range of Values of the Quota)

Let $[q : w_1, w_2, \ldots, w_N]$ be a weighted voting system with $V = w_1 + w_2 + \cdots + w_N$. The **range of values of the quota** is

$$\frac{V}{2} < q \leq V.$$

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What happens in the previous example if $q = 9$? This makes the voting system

$$[9 : 5, 3, 2].$$

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Question

Is this any different from the voting system $[3 : 1, 1, 1]$?

Dictators and Dummies

Example (Dictators)

Eve, Bob, and Alice are all siblings who decide to start a club. Eve, being the oldest, convinces her brother and sister to agree to the voting system $[6 : 6, 2, 2]$ for all club rules.

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Players whose vote does not affect the outcome are **dummies**.

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Definition (Veto Power)

Let $[q : w_1, w_2, \ldots, w_N]$ be a weighted voting system with $V = w_1 + w_2 + \cdots + w_N$. A player with weight w_i has **veto power** if and only if $w_i < q$ and $V - w_i < q$.

Observation: The weights in a weighted voting system are not always an indication of how much **power** a player has. For example: the voting system $[9 : 5, 3, 2]$, each player has equal power.

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Question

How can we determine the power of a player in a weighted voting system?

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Fact

A player P in a winning coalition is a ***critical player*** *for the coalition if and only if $W - w < q$ where W denotes the total weight of the entire coalition and w is the weight of player P.*

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Example

In the weighted voting systems $[4 : 3, 2, 2]$ and $[5 : 3, 2, 2]$ what player has the highest Banzhaf power index?

Example

(a) Find the Banzhaf power distribution of the weighted voting system $[10 : 5, 4, 3, 2, 1]$.

(b) Find the Banzhaf power distribution of the weighted voting system $[11 : 5, 4, 3, 2, 1]$.

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Answer: If there are n players, then there are $2^n - 1$ possible coalitions.

Example (Tie-Breaking Power)

A universities promotion to tenure committee consists of five members. Then dean (D) and four other faculty members (F_1, F_2, F_3, F_4). In this committee, the faculty members all vote first, and motions are carried by simple majority. The dean only votes in the case of a tie.

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Question:

- Is this a weighted voting system?
- If so, what is the Banzhaf power distribution?
- Does the dean have more, less, or equal power as any single member of the committee?

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- BPI for other nations is $\frac{84}{5080}$.

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Let $\{P_1, P_2, P_3\}$ be a coalition where P_2 joins first, then P_1 joins, and finally P_3 joins. The sequential coalition is then $\langle P_2, P_1, P_3 \rangle$.

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- Since we can not choose the same person chosen in the first step, there are now 2 ways to choose the second player.
- After the second player is chosen, there is only one player left to choose.
- Total choices: $3 \cdot 2 \cdot 1 = 3! = 6$ (3 factorial).

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Example

Consider the weighted system $[5 : 4, 3, 2, 1]$. In the coalition $\langle P_2, P_3, P_4 \rangle$, P_3 is the pivotal player.

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- The **Shapely-Shubik power index** for P_i is $\sigma_i = \frac{C_i}{T}$.

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- The **Shapely-Shubik power index** for P_i is $\sigma_i = \frac{C_i}{T}$.
- The **Shapely-Shubik power distribution** is the set $\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$.

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