Chapter 1. The Mathematics of Voting

Introduction to Contemporary Mathematics
Math 112

Example (The Math Club Election)

The math club is electing a new president. The candidates are Alisha (A), Boris (B), Carmen (C), and Dave (D). There were a total of 37 ballots submitted. The raw results are:

Ballot		Ballot		Ballot		Ballot		Ballot	
1^{st}:	A	1^{st}:	C	1^{st}:	D	1^{st}:	B	1^{st}:	C
2^{nd}:	B	2^{nd}:	B	2^{nd}:	C	2^{nd}:	D	2^{nd}:	D
3^{rd}:	C	3^{rd}:	D	3^{rd}:	B	3^{rd}:	C	3^{rd}:	B
4^{th}:	D	4^{th}:	A	4^{th}:	A	4^{th}:	A	4^{th}:	A
Total	14	Total	10	Total	8	Total	4	Total	1

Question: Who should be the winner of the election and why?

Ballots

Definition

- A ballot in which the voters are asked to rank the candidates in order of preference is called a **preference ballot**.
- A ballot in which ties are not allowed is called a **linear ballot**.

Preference Schedule

Example (The Math Club Election)

Data from the ballots can be organized in the following table called a **preference schedule**.

Number of voters	14	10	8	4	1
1^{st} choice	A	C	D	B	C
2^{nd} choice	B	B	C	D	D
3^{rd} choice	C	D	B	C	B
4^{th} choice	D	A	A	A	A

An election can be thought of as a process that takes an *input* (the ballots) and produces an *output* (the winner of the election). The preference schedule is the input, neatly organized to be used for the counting.

Transitivity and Elimination of Candidates

Definition

Voter's preferences are **transitive**. That is a voter who prefers candidate A over candidate B and prefers candidate B over candidate C will automatically prefer candidate A over candidate C.

Observation: If we need to know which candidate a voter would vote for if it came down to a choice between just X and Y, all we have to do is look at where X and Y are placed on that voters's ballot. Whichever is higher would be the one getting the vote.

Question: What happens if Boris drops out of the race for math club president?

1.2. The Plurality Method

Definition

The **plurality method** is a method in which the winner is determined by the candidate (called the **plurality candidate**) with the most first place votes.

Example (The Math Club Election)

Number of voters	14	10	8	4	1
1^{st} choice	A	C	D	B	C
2^{nd} choice	B	B	C	D	D
3^{rd} choice	C	D	B	C	B
4^{th} choice	D	A	A	A	A

The winner using the plurality method is Alisha (A).

Principle of majority rule: In a democratic election between two candidates, the candidate with a majority (more than half) of the votes should be the winner. The candidate with the majority of votes is called the **majority candidate**.

Warning: For a candidate to have a majority, that candidate must have more than half of the votes.

The Majority Criterion

Question

If a candidate has a majority (more than half) of first-place votes, should that candidate be the winner of the election?

The Majority Criterion: If candidate X has a majority of the first-place votes, then candidate X should be the winner of the election.

Question

Under the plurality method, does a majority candidate win the election?

Fact

A majority always implies a plurality. That is: if a candidate has more than half of the votes, then that candidate will automatically have more votes than any other candidate in the election.

Definition

If a voting method declares someone other than the majority candidate the winner, then the voting method **violates** the majority criterion.

Question

If there is more than two options, is the plurality method always a good indicator of who or what should win the election?

Example (The Marching Band Election)

The marching band has invitations to play at the Rose Bowl (R), the Hula Bowl (H), the Fiesta Bowl (F), the Orange Bowl (O), and the Sugar Bowl (S). 100 members of the band vote and the results are in the following preference schedule.

Number of voters	49	48	3
1^{st} choice	R	H	F
2^{nd} choice	H	S	H
3^{rd} choice	F	O	S
4^{th} choice	O	F	O
5^{th} choice	S	R	R

What bowl game should the marching band play at?

The Condorcet Criterion

Definition

A candidate preferred by a majority of the voters over every other candidate when the candidates are compared in head-to-head comparisons is called the **Condorcet candidate**.

12.3: 4,6,8,10,12,13,16,20,23,25,40,42,45,46,48,50. **The Condorcet Criterion:** If candidate X is preferred by the voters over each of the other candidates in a head-to-head comparison, then candidate X should be the winner of the election.

Observation: In the marching band election, the plurality method **violates** the Condorcet criterion.

Insincere Voting

Definition

Let A, B, and C be candidates in an election. An **insincere voter** (or a **strategic voter**) is a voter who prefers candidate A, over candidate B and candidate B over candidate C, but knows that candidate A has no chance of winning the election, so the voter votes for candidate B.

Example (The Marching Band Election)

Number of voters	49	48	3
1st choice	R	H	F
2nd choice	H	S	H
3rd choice	F	O	S
4th choice	O	F	O
5th choice	S	R	R

Assume that the three voters (in the last column) have been talking with their fellow band members and come to the conclusion that the Fiesta bowl is not a very popular choice. So instead of wasting their vote on the Fiesta bowl, they vote for the Hula bowl as their top choice.

Example (The Marching Band Election)

Number of voters	49	48	3
1^{st} choice	R	H	H
2^{nd} choice	H	S	F
3^{rd} choice	F	O	S
4^{th} choice	O	F	O
5^{th} choice	S	R	R

Assume that the three voters (in the last column) have been talking with their fellow band members and come to the conclusion that the Fiesta bowl is not a very popular choice. So instead of wasting their vote on the Fiesta bowl, they vote for the Hula bowl as their top choice.

1.3. The Borda Count Method

Definition (Borda Count Method)

Each place on a (individual) ballot is assigned points. In an election with N candidates, last place is assigned 1 point, second to last place is assigned 2 points, up to N points for first place. After all the points are tallied, the candidate with the most points is the winner (called the **Borda winner**).

Note: With the Borda count method, a tie between two or more candidates is possible. In the case of a tie, there can be a predetermined tie-breaker procedure or ties can be allowed to stand.

Example (The Math Club Election)

Number of voters
1^{st} choice: 4 pts
12.3: 4,6,8,10,12,13,16,20,23,25,40,42,45,46,48,50. 2^{nd} choice: 3 pts
3^{rd} choice: 2 pts
4^{th} choice: 1 pt

Totals:
$A : 56 + 10 + 8 + 4 + 1 = 79$ points
$B : 42 + 30 + 16 + 16 + 2 = 106$ points
$C : 28 + 40 + 24 + 8 + 4 = 104$ points
$D : 14 + 20 + 32 + 12 + 3 = 81$ points
And the Borda winner is Boris!

Example (The School Principal Election)

Mrs. Amaro (A), Mr. Burr (B), Mr. Castro (C), and Mrs. Dunbar (D) are the four finalists for a schools principal. Each of the 11 school board members rank the candidates in the following preference table.

Number of voters	6	2	3
1^{st} choice	A	B	C
2^{nd} choice	B	C	D
3^{rd} choice	C	D	B
4^{th} choice	D	A	A

Who should be the next principal?

Question 1: If the Borda count method is used in the previous example, does it seem fair who the winner is?

Question 2: Does anybody have a majority of the vote?

Question 3: What election method do you think gives the most fair result of this election?

Note: In the previous example the Borda count method fails the majority criterion and the Condorcet criterion.

Note: Although in the previous example the Borda count method may be a poor choice for a election method, for elections with a large amount of voters the Borda count method tends to be a very good and fair method (even though it is still possible to have a result where the option that receives a majority of the votes loses).

1.4. The Plurality-with-Elimination Method (Instant Runoff Voting)

In many elections a majority is required to win the election. With the Plurality method, if there are 3 or more candidates a majority of the votes is less likely.

Question

Is it possible to still use the Plurality method without requiring a runoff election?

Definition (Plurality-with-Elimination Method)

- Count the first-place votes for each candidate. If the plurality winner has a majority, then they are the winner. Otherwise, eliminate the candidate with the fewest first-place votes.
- Count the first-place votes for each candidate (note that candidates on the ballots with the eliminated candidate move up). If the plurality winner has a majority, then they are the winner. Otherwise, repeat this process until the plurality winner has the majority.

Example (The Math Club Election)

Number of voters	14	10	8	4	1
1^{st} choice:	A	C	D	B	C
2^{nd} choice:	B	B	C	D	D
3^{rd} choice:	C	D	B	C	B
4^{th} choice:	D	A	A	A	A

Remove B

Number of voters	14	10	8	4	1
1^{st} choice:	A	C	D	D	C

Remove C

Number of voters	14	10	8	4	1
1^{st} choice:	A	D	D	D	D

Question

Does the Plurality-with-Elimination method satisfy the majority criterion?

Answer: The plurality-with-elimination does satisfy the majority criterion.

Example (Restaurant of the Year)

A magazine is having an election to choose the "Restaurant of the Year." The choices are Andre's (A), Borrelli (B), Casablanca (C), Dante (D), and Escargot (E). The results are organized in the following preference schedule.

Number of voters	8	7	6	2	1
1^{st} choice:	A	D	D	C	E
2^{nd} choice:	B	B	B	A	A
3^{rd} choice:	C	A	E	B	D
4^{th} choice:	D	C	C	D	B
5^{th} choice:	E	E	A	E	C

Total Votes: $8 + 7 + 6 + 2 + 1 = 24$
Dante has a total of $7 + 6 = 13$ votes.Thus Dante has a majority of votes and is therefore the winner.

Question

Does the Plurality-with-Elimination method satisfy the majority criterion?

Answer: The plurality-with-elimination does satisfy the majority criterion.

Question

Does the Plurality-with-Elimination method satisfy the majority criterion?

Answer: The plurality-with-elimination does satisfy the majority criterion.

Example (Next Summer Olympic Games)

Three cities, Athens (A), Barcelona (B), and Calgary, are competing to host the next Summer Olympic Games. The final decision is made by 29 voters using the plurality-with-elimination method. An initial straw poll is taken with the following results.

Number of voters	7	8	10	4
1^{st} choice:	A	B	C	A
2^{nd} choice:	B	C	A	C
3^{rd} choice:	C	A	B	B

The second and final vote yielded the following results.

Number of voters	7	8	10	4
1^{st} choice:	A	B	C	C
2^{nd} choice:	B	C	A	A
3^{rd} choice:	C	A	B	B

Question

- What city wins the straw poll?
- What city actually wins?

Note: The change in the votes from the straw poll to the votes in the final election were in favor of Calgary. Yet Calgary won the straw poll and lost the election.

This example violates the monotonicity criterion.

The Monotonicity Criterion

Definition (The Monotonicity Criterion)

If candidate X is a winner of an election and, in a reelection, the only changes in the ballots are changes that favor X (and only X), then X should remain a winner of the election.

1.5. The Method of Pairwise Comparisons

Note: So far, none of the voting systems we have discussed satisfy the Condorcet criterion.

Definition (Method of Pairwise Comparisons)

- After all the votes are in, each candidate is matched head-to-head against every other candidate.
- Each head-to-head match up is called a **pairwise comparison**.
- In a pairwise comparison between X and Y every vote is assigned to either X or Y.
- The winner of a pairwise comparison is give one point. In the case of a tie, each candidate is given $\frac{1}{2}$ point.
- The winner of the election is the candidate with the most overall points.

Example (The Marching Band Election)

The marching band has invitations to play at the Rose Bowl (R), the Hula Bowl (H), the Fiesta Bowl (F), the Orange Bowl (O), and the Sugar Bowl (S). 100 members of the band vote and the results are in the following preference schedule.

Number of voters	49	48	3
1^{st} choice	R	H	F
2^{nd} choice	H	S	H
3^{rd} choice	F	O	S
4^{th} choice	O	F	O
5^{th} choice	S	R	R

What bowl game should the marching band play at?

R vs H.

Number of voters	49	48	3
1st choice	R	H	F
2nd choice	H	S	H
3rd choice	F	O	S
4th choice	O	F	O
5th choice	S	R	R

- R has 49 votes.
- H has $48 + 3 = 51$ votes.
- H wins and gets 1 point.

R vs H,F,S, or O: Note that R will loose every head-to-head match since it comes in last place in both the second and third columns.

Number of voters	49	48	3
1^{st} choice	R	H	F
2^{nd} choice	H	S	H
3^{rd} choice	F	O	S
4^{th} choice	O	F	O
5^{th} choice	S	R	R

- R has 49 votes.
- other games have $48 + 3 = 51$ votes.
- H,F,S, and O win against R and each get 1 point.

S vs H.

Number of voters	49	48	3
1st choice	R	H	F
2nd choice	H	S	H
3rd choice	F	O	S
4th choice	O	F	O
5th choice	S	R	R

- H has $49 + 48 + 3 = 100$ votes.
- S has 0 votes.
- H wins and gets 1 point.
- H now has 2 points.

Number of voters	49	48	3
1st choice	R	H	F
2nd choice	H	S	H
3rd choice	F	O	S
4th choice	O	F	O
5th choice	S	R	R

In a similar way, we see

- H wins against F
- H wins against O
- Therefore H has a total of 4 points.
- Note that H will have the most points. Why?

Hence if the method of pairwise comparisons is used, the marching band will play at Hula Bowl.

Example (The NFL Draft)

A new NFL expansion team is being developed. The team will be receiving the number-one pick in the next draft. The team has to choose between the following five players: Allen (A), Byers (B), Castillo (C), Dixon (D), and Evans (E). The team executives and coaches all vote and their results are in the following preference schedule.

Number of voters	2	6	4	1	1	4	4
1$^{\text{st}}$ choice	A	B	B	C	C	D	E
2$^{\text{nd}}$ choice	D	A	A	B	D	A	C
3$^{\text{rd}}$ choice	C	C	D	A	A	E	D
4$^{\text{th}}$ choice	B	D	E	D	B	C	B
5$^{\text{th}}$ choice	E	E	C	E	E	B	A

A vs. B:

Number of voters	2	6	4	1	1	4	4
1^{st} choice	A	B	B	C	C	D	E
2^{nd} choice	D	A	A	B	D	A	C
3^{rd} choice	C	C	D	A	A	E	D
4^{th} choice	B	D	E	D	B	C	B
5^{th} choice	E	E	C	E	E	B	A

- A has $2 + 1 + 4 = 7$ votes.
- B has $6 + 4 + 1 + 4 = 15$ votes.
- B wins and gets 1 point.

A vs. C:

Number of voters	2	6	4	1	1	4	4
1st choice	A	B	B	C	C	D	E
2nd choice	D	A	A	B	D	A	C
3rd choice	C	C	D	A	A	E	D
4th choice	B	D	E	D	B	C	B
5th choice	E	E	C	E	E	B	A

- A has $2 + 6 + 4 + 4 = 16$ votes.
- C has $1 + 1 + 4 = 6$ votes.
- A wins and gets 1 point.

A vs. D:

Number of voters	2	6	4	1	1	4	4
1st choice	A	B	B	C	C	D	E
2nd choice	D	A	A	B	D	A	C
3rd choice	C	C	D	A	A	E	D
4th choice	B	D	E	D	B	C	B
5th choice	E	E	C	E	E	B	A

- A has $2 + 6 + 4 + 1 = 13$ votes.
- D has $1 + 4 + 4 = 9$ votes.
- A wins and gets 1 point.
- A now has 2 points.

A vs. E:

Number of voters	2	6	4	1	1	4	4
1st choice	A	B	B	C	C	D	E
2nd choice	D	A	A	B	D	A	C
3rd choice	C	C	D	A	A	E	D
4th choice	B	D	E	D	B	C	B
5th choice	E	E	C	E	E	B	A

- A has $2 + 6 + 4 + 1 + 1 + 4 = 18$ votes.
- E has 4 votes.
- A wins and gets 1 point.
- A now has 3 points.

B vs. C:

Number of voters	2	6	4	1	1	4	4
1st choice	A	B	B	C	C	D	E
2nd choice	D	A	A	B	D	A	C
3rd choice	C	C	D	A	A	E	D
4th choice	B	D	E	D	B	C	B
5th choice	E	E	C	E	E	B	A

- B has $6 + 4 = 10$ votes.
- C has $2 + 1 + 1 + 4 + 4 = 12$ votes.
- C wins and gets 1 point.

B vs. D:

Number of voters	2	6	4	1	1	4	4
1st choice	A	B	B	C	C	D	E
2nd choice	D	A	A	B	D	A	C
3rd choice	C	C	D	A	A	E	D
4th choice	B	D	E	D	B	C	B
5th choice	E	E	C	E	E	B	A

- B has $6 + 4 + 1 = 11$ votes.
- D has $2 + 1 + 4 + 4 = 11$ votes.
- B and D tie and each get $\frac{1}{2}$ point.
- B now has $1\frac{1}{2}$ points and D has $\frac{1}{2}$ point.

B vs. E:

Number of voters	2	6	4	1	1	4	4
1st choice	A	B	B	C	C	D	E
2nd choice	D	A	A	B	D	A	C
3rd choice	C	C	D	A	A	E	D
4th choice	B	D	E	D	B	C	B
5th choice	E	E	C	E	E	B	A

- B has $2 + 6 + 4 + 1 + 1 = 14$ votes.
- E has $4 + 4 = 8$ votes.
- B wins and gets 1 point.
- B now has $2\frac{1}{2}$ points.

C vs. D:

Number of voters	2	6	4	1	1	4	4
1st choice	A	B	B	C	C	D	E
2nd choice	D	A	A	B	D	A	C
3rd choice	C	C	D	A	A	E	D
4th choice	B	D	E	D	B	C	B
5th choice	E	E	C	E	E	B	A

- C has $6 + 1 + 1 + 4 = 12$ votes.
- D has $2 + 4 + 4 = 10$ votes.
- C wins and gets 1 point.
- C now has 2 points.

C vs. E:

Number of voters	2	6	4	1	1	4	4
1st choice	A	B	B	C	C	D	E
2nd choice	D	A	A	B	D	A	C
3rd choice	C	C	D	A	A	E	D
4th choice	B	D	E	D	B	C	B
5th choice	E	E	C	E	E	B	A

- C has $2 + 6 + 1 + 1 = 10$ votes.
- E has $4 + 4 + 4 = 12$ votes.
- E wins and gets 1 point.

D vs. E:

Number of voters	2	6	4	1	1	4	4
1^{st} choice	A	B	B	C	C	D	E
2^{nd} choice	D	A	A	B	D	A	C
3^{rd} choice	C	C	D	A	A	E	D
4^{th} choice	B	D	E	D	B	C	B
5^{th} choice	E	E	C	E	E	B	A

- D has $2 + 6 + 4 + 1 + 1 + 4 + 4 = 22$ votes.
- E has 0 votes.
- D wins and gets 1 point.

Results:

- A has 3 points
- B has $2\frac{1}{2}$ points
- C has 2 points
- D has $1\frac{1}{2}$ points and
- E has 1 point.

A wins.

It is later found out that Castillo (C) has received a schollarship to go to graduate school in mathematics and will not be playing in the NFL.

Question

Will Allen (A) still be the teams first choice?

New Results:

- A has 2 points.
- B has $2\frac{1}{2}$ points.
- D has $1\frac{1}{2}$ points.
- E has 0 points.

Byers (B) is the winner and not Allen (A).

The NFL draft seems to be "unfair" to Allen. This example shows that the method of pairwise comparison can violate the **Independence-of-Irrelevant-Alternatives Criterion**.

Definition (Independence-of-Irrelevant-Alternatives Criterion)

If candidate X is a winner of an election and in a recount election one of the *non-winning* candidates withdraws or is disqualified, then X should still be a winner of the election.

Example (Math Club Election)

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Use pairwise comparisons to determine the winner.

A vs. B:

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

- A has 14 votes
- B has 10+8+4+1 = 23 votes.
- B wins and gets 1 point.

A vs. C or D:

Number of voters	14	10	8	4	1
1^{st} choice	A	C	D	B	C
2^{nd} choice	B	B	C	D	D
3^{rd} choice	C	D	B	C	B
4^{th} choice	D	A	A	A	A

- A has 14 votes
- C or D has $10+8+4+1 = 23$ votes.
- C or D wins and each gets 1 point.

B vs. C:

Number of voters	14	10	8	4	1
1^{st} choice	A	C	D	B	C
2^{nd} choice	B	B	C	D	D
3^{rd} choice	C	D	B	C	B
4^{th} choice	D	A	A	A	A

- B has $14 + 4 = 18$ votes
- C has $10 + 8 + 1 = 19$ votes.
- C wins and gets 1 point.
- C now has 2 points.

B vs. D:

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

- B has $14 + 10 + 4 = 28$ votes
- D has $8 + 1 = 9$ votes.
- B wins and gets 1 point.
- B now has 2 points.

C vs. D:

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

- C has $14 + 10 + 1 = 25$ votes
- C has $8 + 4 = 12$ votes.
- C wins and gets 1 point.
- C now has 3 points.

Results:

- C has 3 points.
- B has 2 points.
- D has 1 point.
- A has 0 points.
- Carmen is the winner!

Finite Sums

Example

What is $1 + 2 + 3 + \cdots + 99 = ?$

Let $S = 1 + 2 + 3 + \cdots + 99$. Then

$$\begin{array}{rcccccccccccc} S & = & 1 & + & 2 & + & 3 & +\cdots+ & 98 & + & 99 \\ +\ S & = & 99 & + & 98 & + & 97 & +\cdots+ & 2 & + & 1 \\ \hline 2S & = & 100 & + & 100 & + & 100 & +\cdots+ & 100 & + & 100 \end{array}$$

Right side $= (99)(100)$. Therefore $2S = (99)(100)$ and thus

$$S = \frac{(99)(100)}{2}.$$

In general: $1 + 2 + \cdots + N = \frac{N(N+1)}{2}$.

Example

What is $1 + 2 + 3 + \cdots + 2016$?

$$N = 2016 \Rightarrow 1 + 2 + \cdots + 2016 = \frac{2016(2017)}{2} = 2,033,136.$$

Example

1. Calculate $1 + 2 + \cdots + 200$.
2. Calculate $1 + 2 + \cdots + 300$.
3. Calculate $201 + 202 + \cdots + 300$.

Question

In a election with N candidates, how many pairwise comparisons are there?

Answer: $\frac{(N-1)N}{2}$.

16. Rankings

In many situations, it is important to know who the second, third, . . . places are. To do this, we need to develop a **ranking** system.

Definition

An **extended ranking method** is done by determining the ranks by extending the results to the other candidates.

Definition

The **recursive ranking system** determines the ranks by eliminating the first place candidate and recalculating the winner to determine second place. Repeat this process until all desired ranks are found.

Example (Math Club Election)

Number of voters	14	10	8	4	1
1st choice	A	C	D	B	C
2nd choice	B	B	C	D	D
3rd choice	C	D	B	C	B
4th choice	D	A	A	A	A

Determine the first second and third place rankings using the plurality method.

Fairness Criteria

- The Majority Criterion
- The Condorcet Criterion
- The Monotonicity Criterion
- The Independence-of-Irrelevant-Alternatives Criterion

- Plurality Method: violates the condorcet criterion.
- The Borda Count Method: violates the majority and the condorcet criteria
- The Plurality-with-Ellimination Method: violates the monotonicity criterian.
- The Method of Pairwise Comparisons: violates the independence-of-irrelevant-alternatives criterion.

Question

Does there exist a voting method that satisfies all of the fairness criteria?

Arrow's Impossibility Theorem

Answer: No!

Theorem (Arrow's Impossibility Theorem)

In an election with 3 or more candidates it is mathematically impossible for a democratic voting method to satisfy all of the fairness criteria.