Statistics
MATH 371
Homework 0
Part 1 of 2.

Exercise 1. Let $A_1, A_2, \ldots$ be an arbitrary infinite sequence of events, and let $B_1, B_2, \ldots$ be another infinite sequence of events defined as

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_1^c \cap A_2 \\ B_3 &= A_1^c \cap A_2^c \cap A_2 \\ &\vdots \end{aligned}$$

Prove that

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \Pr(B_i)$$

for $n = 1, 2, \ldots,$ and that

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(B_i).$$

Exercise 2. Let $A_1, A_2, \ldots$ be an infinite sequence of events such that $A_1 \subset A_2 \subset \cdots$. Prove that

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \Pr(A_n).$$

Hint: Use Exercise 1.

Exercise 3. Let $A_1, A_2, \ldots$ be an infinite sequence of events such that $A_1 \supset A_2 \supset \cdots$. Prove that

$$\Pr\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \Pr(A_n).$$

Exercise 4. A student takes a 5 question true-or-false quiz and, not having studied, randomly guesses the answer to each question. Suppose that a correct answer is worth 1 point and an incorrect answer is worth -1 point. Find the probability that the student

(a) gets 5 points.
(b) gets -5 points.
(c) gets 3 points.
(d) gets 2 points.
(e) gets a positive score.

Exercise 5. For any two events A and B with $\Pr(B) > 0$, prove that $\Pr(A^c|B) = 1 - \Pr(A|B)$.

Exercise 6. Suppose that A, B, and C are three independent events such that $\Pr(A) = \frac{1}{4}$, $\Pr(B) = \frac{1}{3}$, and $\Pr(C) = \frac{1}{2}$.

(a) Determine the probability that none of these three events will occur.
(b) Determine the probability that exactly one of these events will occur.

Exercise 7. Suppose that the pdf of a random variable X is

$$f(x) = \begin{cases} cx^2 & \text{for } 1 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant c and sketch the pdf.
(b) Find the value of $\Pr(X > \frac{3}{2})$.

Exercise 8. Let f be the joint pf/pdf of random variables X and Y, with X discrete and Y continuous. Prove that the marginal pf of X is

$$f_X(x) = \Pr(X = x) = \int_{\mathbb{R}} f(x, y)\, dy \text{ for all } x,$$

and the marginal pdf of Y is

$$f_Y(y) = \sum_x f(x, y), \text{ for } y \in \mathbb{R}.$$