Math 371: Statistics

Exam 1

October 21, 2016

NAME:

To receive full credit you must clearly show all work and justify your answers. No books or calculators are allowed during this exam. You are allowed one page (letter size) of notes. This is a 50 minute exam.

Question:	1	2	3	4	5	6	Total
Points:	10	10	10	10	10	0	50
Score:							

1. (10 points) Consider the function

$$f(x) = \begin{cases} \frac{c}{x} & \text{if } x = 1, 2, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Does there exist a c such that $f(x)$ is a probability function? Explain your answer.

Fact. Let X be a random variable with a continuous distribution and cumulative distribution function $F(x)$ and probability density function $f(x)$. If $E(X)$ exists, then

$$\lim_{x\to\infty} x[1-F(x)] = 0.$$

2. (a) (5 points) Suppose that the random variable X has a continuous distribution with cumulative distribution (cdf) $F(x)$. Suppose also that $\Pr(X \geq 0) = 1$ and that $E(X)$ exists. Show that

$$E(X) = \int_0^\infty [1-F(x)]\,dx.$$

(**Hint:** Use integration by parts and the above Fact.)

(b) (5 points) Let X be the time that a customer spends waiting for service in a queue. Suppose that the cumulative distribution function (cdf) of X is

$$F(x) = \begin{cases} 1-e^{-2x} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected waiting time for service?

3. (10 points) If A and B are independent events such that $\Pr(B) < 1$, prove that $\Pr(A^c|B^c) = \Pr(A^c)$. [**Hint:** $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(AB)$.]

4. Suppose that $X_1, \ldots, X_n$ form a random sample from a Poisson distribution with mean $\theta > 0$ unknown. Suppose also that the prior distribution of θ is the Γ-distribution with parameters $\alpha > 0$ and $\beta > 0$.

   (a) (5 points) Find a function the likelihood function $f_n(\underline{x}, \theta)$ is proportional to.

   (b) (5 points) Given that $X_i = x_i$ for all $i = 1, \ldots, n$, find a function the posterior function $\xi(\theta|\underline{x})$ is proportional to. Conclude that the posterior distribution of θ is also a Γ-distribution.

5. (10 points) Suppose that $X_1, \ldots, X_n$ form a random sample from the distribution for which the probability density function is

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $\theta > 0$ is unknown. Find the maximum likelihood estimator of θ.

6. (10 points (bonus)) Suppose that the random variable X has a continuous distribution with cumulative distribution function $F(x)$ and probability density function $f(x)$. Suppose also that $E(X)$ exists. Prove that

$$\lim_{x\to\infty} x[1 - F(x)] = 0.$$

Hint: First prove $x[1 - F(x)] = x\int_x^\infty f(t)\,dt$. Also, observe that $E(X) = \lim_{x\to\infty}\int_{-\infty}^x f(t)\,dt$.