Introduction to Abstract Mathematics
MATH 310
Homework 9
Due Monday November 30

Please show all your work and justify your answers.

Exercise 1. Use the method of minimum counterexample to prove that $3 \mid (2^{2n} - 1)$ for every positive integer n.

Exercise 2. For the relation $R = \{(x, y) \mid x \leq y\}$ defined on $\mathbb{N}$, what is R^{-1}?

Exercise 3. A relation R is defined on $\mathbb{Z}$ by $x\,R\,y$ if $x \cdot y \geq 0$. Prove or disprove the following:
(a) R is reflexive.
(b) R is symmetric.
(c) R is transitive.

Exercise 4. (a) Let R be the relation defined on $\mathbb{Z}$ by $a\,R\,b$ if $a + b$ is even. Prove that R is an equivalence relation and determine the distinct equivalence classes.
(b) Suppose that "even" is replaced by "odd" in (a). Which of the properties reflexive, symmetric, and transitive does R possess?

Exercise 5. Let $H = \{2^m \mid m \in \mathbb{Z}\}$. A relation on R is defined on the set $\mathbb{Q}^+$ of positive rational numbers by $a\,R\,b$ if $\frac{a}{b} \in H$.
(a) Prove that R is an equivalence relation.
(b) Describe the elements in the equivalence class [3].

Exercise 6. A relation R on a nonempty set A is defined to be **circular** if whenever $x\,R\,y$ and $y\,R\,z$, then $z\,R\,x$ for all $x, y, z \in A$. Prove that a relation R on A is an equivalence relation if and only if R is circular and reflexive.

Exercise 7. Let R be a relation on the set $\mathbb{N}$ by $a\,R\,b$ if either $a \mid 2b$ or $b \mid 2a$. Prove or disprove: R is an equivalence relation.

Exercise 8. Prove or disprove: The union of two equivalence relations on a nonempty set is an equivalence relation.

Exercise 9. A relation R is defined on $\mathbb{Z}$ by $a\,R\,b$ if $2a + 2b \equiv 0 \pmod 4$. Prove that R is an equivalence relation and determine the distinct equivalence classes.

Exercise 10. Let R be a relation defined on $\mathbb{Z}$ by $a\,R\,b$ if $a^2 \equiv b^2 \pmod 5$. Prove that R is an equivalence relation and determine the distinct equivalence classes.