Introduction to Abstract Mathematics
MATH 310
Homework 2
Due Friday 18 September

Please show all your work.

Exercise 1. Let U be a universal set and let A and B be two subsets of U such that $A = B$ and $A \cap B \neq \emptyset$. Draw a Venn diagram for each of the following sets.
(a) $\overline{A \cup B}$
(b) $\overline{A} \cap \overline{B}$
(c) $\overline{A \cap B}$
(d) $\overline{A} \cup \overline{B}$.

Exercise 2. Let $A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$.
(a) Determine which of the following are elements of $A : \emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$.
(b) Determine $|A|$.
(c) Determine which of the following are subsets of $A : \emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}$.
(d) Determine $\emptyset \cap A$.
(e) Determine $\{\emptyset\} \cap A$.
(f) Determine $\{\emptyset, \{\emptyset\}\} \cap A$.
(g) Determine $\emptyset \cup A$.
(h) Determine $\{\emptyset\} \cup A$.
(i) Determine $\{\emptyset, \{\emptyset\}\} \cup A$.

Exercise 3. Let $A = \{x \in \mathbb{R} \mid |x - 1| \leq 2\}$, $B = \{x \in \mathbb{R} \mid |x| \geq 1\}$, and $C = \{x \in \mathbb{R} \mid |x + 2| \leq 3\}$.
(a) Express A, B, and C using interval notation.
(b) Determine each of the following sets using interval notation: $A \cup B$, $A \cap B$, $B \cap C$, and $B \setminus C$.

Exercise 4. For a real number r, define $A_r = \{r^2\}$, B_r as the closed interval $[r - 1, r + 1]$ and C_r as the interval $(0, \infty)$. For $S = \{1, 2, 4\}$, determine:
(a) $\bigcup_{\alpha \in S} A_\alpha$ and $\bigcap_{\alpha \in S} A_\alpha$
(b) $\bigcup_{\alpha \in S} B_\alpha$ and $\bigcap_{\alpha \in S} B_\alpha$
(c) $\bigcup_{\alpha \in S} C_\alpha$ and $\bigcap_{\alpha \in S} C_\alpha$.

Exercise 5. For each of the following, find an indexed collection $\{A_n\}_{n \in \mathbb{N}}$ of distinct sets (that is, no two sets are equal) satisfying the given conditions.
(a) $\bigcap_{n=1}^{\infty} A_n = \{0\}$ and $\bigcup_{n=1}^{\infty} A_n = [0, 1]$
(b) $\bigcap_{n=1}^{\infty} A_n = \{-1, 0, 1\}$ and $\bigcup_{n=1}^{\infty} A_n = \mathbb{Z}$.

Exercise 6. For $n \in \mathbb{N}$, let $A_n = (-\frac{1}{n}, 2 - \frac{1}{n})$. Determine $\bigcup_{n \in \mathbb{N}} A_n$ and $\bigcap_{n \in \mathbb{N}} A_n$.

Exercise 7. Let $A = \{1, 2, 3, 4, 5, 6\}$. Give an example of a partition $\mathcal{S}$ of A such that $|\mathcal{S}| = 3$.

Exercise 8. Give an example of a partition of $\mathbb{C}$ into two subsets.

Exercise 9. For $A = \{\emptyset, \{\emptyset\}\}$, determine $A \times \mathcal{P}(A)$ and $|\mathcal{P}(A \times \mathcal{P}(A))|$.

Exercise 10. Let I denote the interval $[0, \infty)$. For each $r \in I$, define

$$A_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 = r^2\}$$
$$B_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 \leq r^2\}$$
$$C_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid x^2 + y^2 > r^2\}$$

(a) Determine $\bigcup_{r \in I} A_r$ and $\bigcap_{r \in I} A_r$.
(b) Determine $\bigcup_{r \in I} B_r$ and $\bigcap_{r \in I} B_r$.
(c) Determine $\bigcup_{r \in I} C_r$ and $\bigcap_{r \in I} C_r$.