AUSLANDER'S THEOREM AND n-ISOLATED SINGULARITIES

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ABSTRACT. One of the most stunning results in the representation theory of Cohen-Macaulay rings is Auslander's well known theorem that states a CM local ring of finite CM type can have at most an isolated singularity. There have been some generalizations of this in the direction of countable CM type by Huneke and Leuschke. In this paper, we focus on a different generalization by restricting the class of modules. Here we consider modules which are high syzygies of MCM modules over non-commutative rings, exploiting the fact that noncommutative rings allow for finer homological behavior. We then generalize Auslander's Theorem in the setting of complete Gorenstein local domains by examining path algebras, which preserve finiteness of global dimension.

1. Introduction. One main focus of the study of representation theory of commutative Noetherian rings is the question of finite Cohen-Macaulay (CM) type–i.e., when a local commutative Noetherian ring R has only finitely many (up to isomorphism) indecomposable maximal Cohen-Macaulay modules. Auslander showed that a complete Cohen-Macaulay local ring R of finite CM type has at most an isolated singularity; that is,

$$\operatorname{gldim} R_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$$

for all non-maximal prime ideals $\mathfrak{p} \in \operatorname{Spec} R$, [1]. Wiegand [16] and Leuschke-Wiegand [10] then proved that finite CM type ascends to and descends from the completion of an excellent local ring R, thus generalizing the theorem to all excellent CM local rings. Finally, Huneke-Leuschke gave a completion-free proof for arbitrary CM local rings in [7]. In the paper of Huneke-Leuschke, the idea of countable CM type is addressed, and they are able to show that if a CM local ring has countable CM type then the singular locus is at most one-dimensional. In this paper we are interested in a different generalization of Auslander's theorem. We wish to restrict our finiteness assumption to a smaller class of modules. To do so we will consider noncommutative algebras over which high-syzygies exhibit similar behavior to maximal Cohen-Macaulay modules over CM local rings. Such algebras will be called n-canonical orders, and Section 4 will focus on their properties. The main result, Theorem 5.1, is that for an n-canonical R-order Λ, if there are only finitely many (up to isomorphism) indecomposable n^{th} syzygies of MCM Λ-modules, then $\operatorname{gldim} \Lambda_{\mathfrak{p}} = n + \dim R_{\mathfrak{p}}$ for all non-maximal prime ideals $\mathfrak{p} \in \operatorname{Spec} R$.

Finally, in section 6 we refocus on the case of commutative rings, using our main theorem to show that if R is a complete Gorenstein local domain and Q is an acyclic quiver such that RQ has finitely many indecomposable first syzygies (of MCM RQ-modules), then R has at most an isolated singularity. This is a generalization of Auslander's theorem for Gorenstein domains.

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2. Background and Notation. Here we will briefly remind the reader of the notation, conventions, and definitions which are heavily utilized in this article. Throughout, R will be a commutative Noetherian ring of finite Krull dimension d. We use the notation $(R,\mathfrak{m},k)$ to imply R is a commutative local Noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = k$. For the convenience of the reader, below we include definitions of the preliminary notions we will use.

Definition 2.1. *Let $(R,\mathfrak{m},k)$ be a commutative local Noetherian ring.*

- *An R-algebra Λ is an R-*order *if it is a MCM R-module.*
- *Denote by* $\operatorname{Mod}\Lambda$ *the category of left Λ-modules and* $\operatorname{mod}\Lambda$ *the full subcategory of* $\operatorname{Mod}\Lambda$ *consisting of finitely generated modules. Unless specified otherwise, when we say M is a Λ-module, we always mean a finitely generated left Λ-module.*
- *Let M be a finitely generated R-module. We say M is* maximal Cohen-Macaulay *if*
$$\operatorname{depth}(M) := \min\{i \in \mathbb{N} \mid \operatorname{Ext}^i_R(k,M) \neq 0\} = d.$$
- *We denote by* $\operatorname{CM}\Lambda$ *the full subcategory of* $\operatorname{mod}\Lambda$ *consisting of modules which are maximal Cohen-Macaulay R-modules.*
- *For a (possibly non-commutative) ring Γ, we will denote by* Γ^{op} *the opposite ring. If M is an abelian group with a right Γ-module structure, we will say $M \in \operatorname{mod}\Gamma^{\mathrm{op}}$ to indicate that M is a left Γ^{op}-module.*
- Λ *is* non-singular *if* $\operatorname{gldim}(\Lambda_\mathfrak{p}) = \dim R_\mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$.
- *We say an order Λ is an* isolated singularity *if* $\operatorname{gldim}(\Lambda_\mathfrak{p}) = \dim R_\mathfrak{p}$ *for all non-maximal prime ideals $\mathfrak{p}$ of R.*
- *For any ring Γ, we denote by* $\operatorname{Proj}\Gamma$ *the full subcategory of* $\operatorname{mod}\Lambda$ *consisting of all projective Γ-modules.*
- *For any module M,* $\operatorname{add} M$ *denotes the* additive closure *of M, i.e., the full subcategory of* $\operatorname{Mod}\Lambda$ *consisting of all modules which are isomorphic to direct summands of finite direct sums of copies of M.*

In the case that R is Cohen-Macaulay with a canonical module ω_R, an R-order possesses a special module akin to ω_R. For details on Cohen-Macaulay rings or canonical modules, see [5, Section 3.3].

Definition 2.2. *Let R be a Cohen-Macaulay ring with canonical module ω_R and Λ an R-order. Then the* canonical module *of Λ is $\omega_\Lambda = \operatorname{Hom}_R(\Lambda, \omega_R)$. We see that ω_Λ is both a $\Lambda-$ and Λ^{op}-module. If ω_Λ is projective as a left Λ-module, then Λ is called a* Gorenstein order.

In the rest of this article, R is always assumed to be a Cohen-Macaulay local ring with canonical module ω_R. We need several functors to study orders. Let Λ be an R-order. We have the following functors.

- The *canonical dual* $D_d(-) := \mathrm{Hom}_R(-,\omega_R) : \mathrm{CM}\,\Lambda \longrightarrow \mathrm{CM}\,\Lambda^{\mathrm{op}}$. Note, this functor is exact on $\mathrm{CM}\,\Lambda$ since $\mathrm{Ext}^i_R(M,\omega_R) = 0$ for $i > 0$ and M an MCM R-module.
- The *Matlis dual* $D := \mathrm{Hom}_R(-,E)$ where E is the injective hull of the residue field, k, of R. Letting $\mathrm{f.l.}R$ denote the full subcategory of $\mathrm{mod}R$ consisting of finite length R-modules, $D : \mathrm{f.l.}R \longrightarrow \mathrm{f.l.}R$ is a duality.
- The functor $(-)^* := \mathrm{Hom}_\Lambda(-,\Lambda) : \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,\Lambda$ which gives a duality $(-)^* : \mathrm{add}\,\Lambda \longrightarrow \mathrm{add}\,\Lambda^{\mathrm{op}}$.
- The *transpose duality* $\mathrm{Tr} : \underline{\mathrm{mod}}\Lambda \longrightarrow \underline{\mathrm{mod}}\Lambda$ given by $\mathrm{Tr}\,M = \mathrm{cok}\,f_1^*$, where $P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$ is a minimal projective resolution of M.
- Finally, we denote $\mathrm{Hom}_R(-,R) = (-)^\dagger$. In the case when R is Gorenstein, we note that $D_d(-) = (-)^\dagger$.

3. Projective Dimension and the Canonical module. In this section we examine orders which exhibit similar behavior as seen in commutative rings. Specifically, we note that by the Auslander-Buchsbaum formula [3], maximal Cohen-Macaulay modules over commutative rings are either projective or have infinite projective dimension. We prove that for orders over CM rings, finite projective dimension of the canonical modules gives a similar result for high syzygies.

A great deal of work has been done to study Gorenstein orders, see e.g., [8] and [9]. These are natural candidates–in the case where R is a Cohen-Macaulay (CM) local ring–for noncommutative crepant resolutions. One reason that Gorenstein orders are so useful is that they exhibit some similar behavior to commutative rings. In particular, they satisfy an Auslander-Buschbaum theorem.

Lemma 3.1. [9, Lemma 2.16] *Let Λ be a Gorenstein R-order. Then for any $X \in \mathrm{mod}\,\Lambda$ with $\mathrm{projdim}_\Lambda X < \infty$ we have*

$$\mathrm{projdim}_\Lambda X + \mathrm{depth}_R X = \dim R.$$

The above result is a special case of the main result of this section, which relates the projective dimension of ω_Λ to the possible projective dimension of all finitely generated Λ-modules.

Theorem 3.2. *Let Λ be an R-order with $\mathrm{projdim}_{\Lambda^{\mathrm{op}}}\,\omega_\Lambda = n$. For any $X \in \mathrm{mod}\,\Lambda$ with $\mathrm{projdim}_\Lambda X < \infty$ we have*

$$\dim R \leqslant \mathrm{projdim}_\Lambda X + \mathrm{depth}_R X \leqslant \dim R + n. \tag{3.1}$$

The following useful result is an application of Theorem 3.2. The proof is left to the reader.

Corollary 3.3. *Let Λ be an R-order. If $\mathrm{gldim}\,\Lambda = n + d$, then $\mathrm{projdim}_{\Lambda^{\mathrm{op}}}\,\omega_\Lambda = n$.*

The rest of this paper is dedicated to orders with $\operatorname{projdim}_{\Lambda^{op}} \omega_\Lambda \leqslant n$. As such, we give this condition a name.

Definition 3.4. *Let R be a CM local ring with canonical module ω. Let Λ be an R-order. We call Λ n-*canonical *if* $\operatorname{projdim}_{\Lambda^{op}} \omega_\Lambda = n$. *A* Gorenstein *order is a 0-canonical order.*

Notation 3.5. *Denote by $\mathcal{S}$ the additive closure of the full subcategory of* $\operatorname{mod}\Lambda$ *consisting of n^{th} syzygies of maximal Cohen-Macaulay Λ-modules, i.e., $\mathcal{S} = \operatorname{add}\{\Omega^n \operatorname{CM}(\Lambda)\}$, where*

$$\Omega^n \operatorname{CM}(\Lambda) := \{M \in \operatorname{Mod}\Lambda \mid M \cong \Omega^n X \text{ for some } X \in \operatorname{CM}\Lambda\}.$$

Now we move on to prove Theorem 3.2, which follows from the Ext-vanishing imposed by being n-canonical.

Lemma 3.6. *Suppose Λ is an n-canonical order over a CM local ring R with canonical module ω. If $M \in \operatorname{CM}\Lambda$ then $\operatorname{Ext}^i_\Lambda(M,\Lambda) = 0$ for $i > n$. In particular, if $X \in \mathcal{S}$, then $\operatorname{Ext}^i_\Lambda(X,\Lambda) = 0$ for $i > 0$.*

Proof. Begin by taking a projective resolution of ω_Λ over Λ^{op}

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \dots \longrightarrow P_0 \longrightarrow \omega_\Lambda \longrightarrow 0$$

and apply $D_d(-)$ to get a resolution

$$0 \longrightarrow \Lambda \longrightarrow I_0 \longrightarrow \dots \longrightarrow I_{n-1} \longrightarrow I_n \longrightarrow 0$$

with $I_j \in \operatorname{add}\omega_\Lambda$. Since $\operatorname{Ext}^i_\Lambda(N,\omega_\Lambda) = 0$ for $i > 0$ and $N \in \operatorname{CM}\Lambda$, we have $\operatorname{Ext}^{n+i}_\Lambda(N,\Lambda) = 0$. The final statement then follows by dimension shifting for Ext.

∎

Proof of Theorem 3.2. First we show that if $X \in \operatorname{CM}\Lambda$ satisfies $\operatorname{projdim}_\Lambda X < \infty$, then $\operatorname{projdim}_\Lambda X \leqslant n$. By Lemma 3.6, if $X \in \operatorname{CM}\Lambda$, then $\operatorname{Ext}^i_\Lambda(X,\Lambda) = 0$ for $i > n$. Since $\operatorname{Ext}^r_\Lambda(X,\Lambda) \neq 0$ for $r = \operatorname{projdim}_\Lambda X$, we must have either $\operatorname{projdim}_\Lambda X \leqslant n$ or $\operatorname{projdim}_\Lambda X = \infty$.

Now, for any module $X \in \operatorname{mod}\Lambda$ with $\operatorname{depth}_R X = t$, the $(d-t)^{th}$ syzygy must be in $\operatorname{CM}\Lambda$ by the Depth Lemma for R and the fact that Λ is MCM over R. We then have

$$\operatorname{projdim}_\Lambda X = (d-t) + \operatorname{projdim}\Omega^{d-t} X \leqslant d - t + n = \dim R - \operatorname{depth}_R X + n.$$

The upper bound of (3.1) follows at once from this. To prove the lower bound we simply note that projective Λ-modules are in $\operatorname{CM}\Lambda$. By the Depth Lemma again, if $\operatorname{depth}_R X = t$, then the first syzygy which could be projective is the $(d-t)^{th}$, as each syzygy can go up in depth by at most 1. Thus

$$\operatorname{projdim}_\Lambda X \geqslant d - \operatorname{depth}_R X.$$

This concludes the proof. ∎

Remark 3.7. *Note that the right inequality of Theorem 3.2 cannot be strengthened to an equality. Indeed, suppose* $M \in \operatorname{CM}\Lambda$ *has* $\operatorname{projdim}_\Lambda M = n$*. Then of course* $\Omega M \in \operatorname{CM}\Lambda$ *has* $\operatorname{projdim}_\Lambda \Omega M = n-1$*, but* $\operatorname{depth}_R \Omega M = \operatorname{depth}_R M$.

It is well known that commutative rings of finite Krull dimension d have either infinite global dimension or finite global dimension equal to d. Thus, for $1 \leqslant n < \infty$, we know commutative n-canonical orders cannot exist. In the noncommutative case we begin by establishing their existence. The following Proposition is clear from Corollary 3.3.

Proposition 3.8. *Let R be a d-dimensional CM local ring with a canonical module. If Λ is an R-order with* $\operatorname{gldim}\Lambda = n+d$*, then Λ is n-canonical.*

Note that we needn't worry about $n < 0$. For a local CM ring R and an order Λ, $\Lambda/\mathfrak{m}\Lambda$ is of fintie length (depth 0) and thus the Depth Lemma for R implies $\operatorname{projdim}_\Lambda \Lambda/\mathfrak{m}\Lambda \geqslant d$. In other words, for an order Λ over a d-dimensional CM local ring, $\operatorname{gldim}\Lambda \geqslant d$.

Example 3.9. *Let k be an infinite field and let R be the complete (2,1)-scroll, that is, $R = k[[x,y,z,u,v]]/I$ with I the ideal generated by the 2×2 minors of $\left(\begin{smallmatrix} x & y & u \\ y & z & v \end{smallmatrix}\right)$. Then, R is a 3-dimensional CM normal domain of finite CM type* [17, 16.12]*. It is known $\Gamma = \operatorname{End}_R(R \oplus \omega)$ is MCM over R, and Γ is symmetric since it is an endomorphism ring over a normal domain* [9, Lemma 2.10]*. But, Smith and Quarles have shown* $\operatorname{gldim}(\Gamma) = 4$ [15] *while* $\dim R = 3$*. Thus Γ is a 1-canonical order.*

We now establish the existence of n-canonical orders for $n \geqslant 1$ with infinite global dimension. To start, we show that the n-canonical property is additive under tensoring.

Lemma 3.10. *Let Λ_1 and Λ_2 be algebras over a Gorenstein local ring R such that Λ_1 and Λ_2 are free R-modules. Then $\Lambda_1 \otimes_R \Lambda_2$ is an R-order and $\omega_{\Lambda_1\otimes\Lambda_2} \cong \omega_{\Lambda_1} \otimes_R \omega_{\Lambda_2}$ as both $\Lambda_1 \otimes_R \Lambda_2$- and $(\Lambda_1 \otimes_R \Lambda_2)^{\mathrm{op}}$-modules.*

Proof. It is not hard to show that if M and N are free R-modules, we have an R-isomoprhism

$$\operatorname{Hom}_R(M,R) \otimes \operatorname{Hom}_R(N,R) \cong \operatorname{Hom}_R(M \otimes_R N, R).$$

Since Λ_1 and Λ_2 are free R-modules and R is Gorenstein (hence $R \cong \omega_R$), this implies that there is an R-isomorphism

$$\begin{aligned}\omega_{\Lambda_1\otimes\Lambda_2} &= \operatorname{Hom}_R(\Lambda_1 \otimes_R \Lambda_2, R)\\ &\cong \operatorname{Hom}_R(\Lambda_1, R) \otimes_R \operatorname{Hom}_R(\Lambda_2, R)\\ &= \omega_{\Lambda_1} \otimes \omega_{\Lambda_2}.\end{aligned}$$

Let

$$\Phi : \operatorname{Hom}_R(\Lambda_1, R) \otimes_R \operatorname{Hom}_R(\Lambda_2, R) \longrightarrow \operatorname{Hom}_R(\Lambda_1 \otimes_R \Lambda_2, R)$$

be the R-isomorphism above, i.e., $[\Phi(f \otimes g)](a \otimes b) = f(a)g(b)$. We must only show this is a morphism of $\Lambda_1 \otimes_R \Lambda_2$-modules. We know any map $F \in \operatorname{Hom}_R(\Lambda_1 \otimes \Lambda_2, \omega)$ is actually $F = \sum_{i=1}^n f_i \otimes g_i$, for $f_i \in \operatorname{Hom}_R(\Lambda_1, \omega)$ and $g_i \in \operatorname{Hom}_R(\Lambda_2, \omega)$. Without loss of generality we may assume $n = 1$. We compute, for any $\lambda_1 \otimes \lambda_2 \in \Lambda_1 \otimes \Lambda_2$,

$$\begin{aligned}[(\lambda_1 \otimes \lambda_2) \cdot \Phi(f \otimes g)](\eta_1 \otimes \eta_2) &= \Phi[f \otimes g]((\eta_1 \otimes \eta_2) \cdot (\lambda_1 \otimes \lambda_2))\\ &= f(\eta_1\lambda_1)g(\eta_2\lambda_2).\end{aligned}$$

On the other hand, $\Phi(f \otimes g)$ is the composition of $f \otimes g$ followed by multiplication $\mu : R \otimes R \longrightarrow R$. We see then

$$\begin{aligned}[\Phi((\lambda_1 \otimes \lambda_2) \cdot f \otimes g))](\eta_1 \otimes \eta_2) &= \mu \circ [(\lambda_1 \otimes \lambda_2) \cdot f \otimes g]((\eta_1 \otimes \eta_2))\\ &= \mu \circ f \otimes g((\eta_1 \otimes \eta_2)(\lambda_1 \otimes \lambda_2))\\ &= \mu \circ f \otimes g(\eta_1\lambda_1 \otimes \eta_1\lambda_1)\\ &= f(\eta_1\lambda_1)g(\eta_1\lambda_1).\end{aligned}$$

Checking that it is a $(\Lambda_1 \otimes_R \Lambda_2)^{\mathrm{op}} = \Lambda_1^{\mathrm{op}} \otimes_R \Lambda_2^{\mathrm{op}}$-module morphism is similar. □

Since we are now able to find the canonical module of orders which are free over R, we get the following examples for R a regular local ring.

Theorem 3.11. *Let $(R, \mathfrak{m}, k)$ be a regular local ring. Suppose Λ_1, Λ_2 are n_1-canonical and n_2-canonical R-orders,respectively. Then $\Lambda_1 \otimes \Lambda_2$ is an $(n_1 + n_2)$-canonical R-order.*

Proof. Since Λ_1 and Λ_2 are MCM over R, and R is a regular local ring, then in fact they are free. Then, noting that $(\Lambda_1 \otimes_R \Lambda_2)^{\mathrm{op}} \cong \Lambda_1^{\mathrm{op}} \otimes_R \Lambda_2^{\mathrm{op}}$, this follows immediately from Lemma 3.10 and [6, Corollary IX.2.7], namely that

$$\begin{aligned}\operatorname{projdim}_{\Lambda_1^{\mathrm{op}}\otimes\Lambda_2^{\mathrm{op}}}(\omega_{\Lambda_1\otimes\Lambda_2}) &= \operatorname{projdim}_{\Lambda_1^{\mathrm{op}}\otimes\Lambda_2^{\mathrm{op}}} \omega_{\Lambda_1} \otimes \omega_{\Lambda_2}\\ &= \operatorname{projdim}_{\Lambda_1^{\mathrm{op}}} \omega_{\Lambda_1} + \operatorname{projdim}_{\Lambda_2^{\mathrm{op}}} \omega_{\Lambda_2}.\end{aligned}$$

□

With this in hand, we can prove the existence of orders which are n-canonical and have infinite global dimension. We first remind the reader of some basics on path algebras.

3.1. Homological behavior of Path Algebras. The main theorem of Chapter 3 is homological in nature. As such, we collect some background on the homological behavior of path algebras.

Definition 3.12. *A* quiver $Q = (Q_0, Q_1, s, t)$ *is a directed graph Q with vertex set Q_0 and arrow set Q_1. There are two maps $s, t : Q_1 \longrightarrow Q_0$ where for an arrow $e \in Q_1$, $s(e)$ is the origin of e and $t(e)$ is the destination of e. A* path *in Q is a sequence of arrows $a_n a_{n-1} \dots a_1$ such that $t(a_i) = s(a_{i+1})$ for $1 \leqslant i \leqslant n-1$.*

Definition 3.13. *Let R be a commutative Noetherian ring and Q a quiver. The* path algebra *RQ of Q over R is the free module on the basis the set of all paths $a_l a_{l-1} ... a_1$ of length $l \geqslant 0$ in Q. The product of two basis vectors (i.e., paths) $b_k ... b_1$ and $a_l ... a_1$ of RQ is defined by*

$$(b_k ... b_1) \cdot (a_l ... a_1) = b_k ... b_1 a_l ... a_1$$

if $t(a_l) = s(b_1)$ and 0 otherwise, i.e., the product of arrows $b \cdot a$ is nonzero if and only if b leaves the vertex where a arrives. Multiplication is extended to linear combinations of basis elements R-linearly.

The next well-known result is what makes path algebras a convenient choice for relating global dimension information about orders back to the commutative base rings, for details see e.g. [14, Chapter 2].

Proposition 3.14. *Let Q be a quiver without oriented cycles. Let R be a regular local ring of dimension d and RQ the path algebra of Q over R. Then,* $\operatorname{gldim} RQ = d + 1$. *If R is not regular, then* $\operatorname{gldim} RQ = \infty$.

We need one more convenient Lemma in order to work with path algebras efficiently.

Lemma 3.15. *Let R be an algebra over a commutative local ring T. Let Q a quiver, and I a right ideal in TQ. Then there is an isomorphism of T-algebras*

$$RQ/_{IRQ} \cong TQ/_{I} \otimes_T R.$$

Proof. We begin with the case that $I = 0$. Define a map $\Phi : TQ \times T \longrightarrow TQ$ via $\Phi(p, r) = rp$ for a path p and extending linearly. This map is clearly T-bilinear, and hence induces a map $\Phi : TQ \otimes_T R \longrightarrow RQ$. This map is onto since any basis element of RQ (i.e., a path in Q) say p, is $\Phi(p \otimes 1)$. We note that any element of $TQ \otimes R$ can be written as $\sum_{i=1}^{n} p_i \otimes s_i$ for paths p_i. Now, if

$$\Phi\left(\sum_{i=1}^{n} (p_i \otimes s_i)\right) = s_1 p_1 + s_2 p_2 + \cdots + s_n p_n = 0,$$

it must be that $s_i = 0$ for all i, since the paths form a basis over R for RQ.

We now move to the case that I is a non-zero ideal. We note

$$T \otimes_T TQ/I \cong R \otimes_T TQ \otimes_{TQ} TQ/I \cong RQ \otimes_{TQ} TQ/I \cong RQ/IRQ,$$

where the second isomorphism follows from the $I = 0$ case. $\square$

And now we can produce a natural example of n-canonical orders with infinite global dimension.

Theorem 3.16. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional Gorenstein local domain. Suppose Q is an acyclic quiver. Then $\Lambda = RQ$ is a 1-canonical R-order. If R is not regular, then* $\operatorname{gldim} \Lambda = \infty$.

For the proof we will need to reduce to the case where R is complete via the following lemma.

Lemma 3.17. *Suppose R is a CM local ring with a canonical module ω_R and that $R \hookrightarrow S$ is a faithfully flat (commutative) ring extension such that $\dim S = \dim R$ and S has a canonical module $\omega_S = \omega_R \otimes_R S$ (e.g., if $S = \widehat{R}$). Let Λ be an R-order. We have that Λ is an n-canonical R-order if and only if $\Lambda \otimes_R S$ is an n-canonical S-order.*

Proof. We really only need to prove two facts. First we note that since S is faithfully flat

$$\operatorname{Hom}_R(M, N) \otimes_R S \cong \operatorname{Hom}_S(M \otimes_R S, N \otimes_R S).$$

It follows at once that $\omega_{\Lambda \otimes_R S} \cong \omega_\Lambda \otimes_R S$ over S. Verifying that this is a $\Lambda \otimes_R S$-isomorphism is straightforward. Next, since exactness of Λ-module sequences can be checked as R-modules, S is faithfully flat over R, and $- \otimes_R S$ takes projective Λ-modules to projective $\Lambda \otimes_R S$-modules, we see that

$$\operatorname{projdim}_\Lambda \omega_\Lambda = \operatorname{projdim}_{\Lambda \otimes_R S} \omega_{\Lambda \otimes_R S}.$$

The lemma follows at once from these two observations.

$\square$

Proof of Theorem 3.16. We reduce to the case where R is complete. Let $\widehat{R}$ denote the completion of R with respect to the maximal ideal. By Lemma 3.17, we see that RQ is 1-canonical if and only if $RQ \otimes_R \widehat{R}$ is 1-canonical. But, by Lemma 3.15, we know that $RQ \otimes \widehat{R} \cong \widehat{R}Q$. Thus we see RQ is 1-canonical if and only if $\widehat{R}Q$ is 1-canonical. Thus we may assume R is complete.

Now, by Cohen's Structure Theorem for complete local rings, [12, Theorem 8.24], R is an order over some d-dimensional regular local ring S. Since R is a Gorenstein local ring and an order over S, we have $R \cong \omega_R \cong \operatorname{Hom}_S(R, S)$ and $\operatorname{projdim}_S \omega_R = 0$ since R is MCM over S and hence free; i.e., R is a 0-canonical S-order. Further, by Proposition 3.14, we know that $\operatorname{gldim} SQ = d + 1$ and hence by Theorem 3.2, $\operatorname{projdim}_{SQ^{\mathrm{op}}} \omega_{SQ} = 1$; i.e., SQ is a 1-canonical S-order. Now, by Proposition 3.15 and Corollary 3.3, $\Lambda := RQ \cong R \otimes_S SQ$ is a 1-canonical R-order.

All that is left is to establish that Λ is in fact an R-order (indeed, it is R-free) and that $\operatorname{Hom}_S(\Lambda, S) \cong \operatorname{Hom}_R(\Lambda, R)$, i.e. that the canonical module of Λ as an R-order agrees with that as an S-order. For the final assertion, we see

$$\operatorname{Hom}_R(\Lambda, R) \cong \operatorname{Hom}_R(\Lambda, \operatorname{Hom}_S(R, S)) \cong \operatorname{Hom}_S(\Lambda \otimes_R R, S) \cong \operatorname{Hom}_S(\Lambda, S).$$

It is straight-forward to verify this is also an isomorphism of Λ-modules. Lastly, by Lemma 3.14 we know that if R is not regular, we have $\operatorname{gldim} \Lambda = \infty$. □

4. Higher Isolated Singularities. The main theorem of this paper is that if an order Λ is n-canonical and has only finitely many nonsiomorphic indecomposable modules in $\Omega^n \operatorname{CM} \Lambda$, then Λ has finite global dimension on the punctured spectrum of R. In this section we show that over orders with this property, high syzygies behave much like MCM modules over isolated singularities.

Definition 4.1. *Let Λ be an order over a CM ring R. We call Λ an n-*isolated singularity *if*

$$\operatorname{gldim} \Lambda_{\mathfrak{p}} \leqslant n + \dim R_{\mathfrak{p}}$$

*for all non-maximal prime ideals $\mathfrak{p}$. We say Λ is n-*nonsingular *if* $\operatorname{gldim} \Lambda_{\mathfrak{p}} = n + \dim R_{\mathfrak{p}}$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$.

Remark 4.2. *It follows from the definition that if Λ is an n-isolated singularity, it is also an m-isolated singularity for any $m \geqslant n$. It might be interesting to study "strict" n-isolated singularities where* $\operatorname{gldim} \Lambda_{\mathfrak{p}} = n + \dim R_{\mathfrak{p}}$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$*, as well as non-strict ones.*

When Λ is a isolated singularity (the $n = 0$ case) it is known that all modules in $\operatorname{CM} \Lambda$ are d^{th} syzygies, (see [11, Corollary A.15] for a proof in the commutative case, the proof for Λ is similar), but this is not true in general. Over n-isolated singularities, we have the following result, which again allows us to bound the projective dimension of modules in $\operatorname{CM} \Lambda$ on the punctured spectrum.

Lemma 4.3. *Let Λ be an n-isolated singularity over a CM local ring R. Then if $M \in CM(\Lambda)$ we have*

$$\operatorname{projdim} M_{\mathfrak{p}} \leqslant n$$

for all non-maximal primes $\mathfrak{p}$.

Proof. Let $M \in \operatorname{CM} \Lambda$ and $\mathfrak{p} \in \operatorname{Spec} R$. It follows that $M_{\mathfrak{p}} \in \operatorname{CM} \Lambda_{\mathfrak{p}}$. Pick a maximal $M_{\mathfrak{p}}$-regular sequence $x_1, \ldots, x_t \in \mathfrak{p} R_{\mathfrak{p}}$. We have an exact sequence

$$0 \longrightarrow M_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}}/x_1 M_{\mathfrak{p}} \longrightarrow 0$$

which induces an exact sequence

$$\operatorname{Ext}^{i-1}_{\Lambda_{\mathfrak{p}}}(M_{\mathfrak{p}}, -) \longrightarrow \operatorname{Ext}^{i}_{\Lambda_{\mathfrak{p}}}(M_{\mathfrak{p}}/x_1 M_{\mathfrak{p}}, -) \longrightarrow \operatorname{Ext}^{i}_{\Lambda_{\mathfrak{p}}}(M_{\mathfrak{p}}, -) \longrightarrow \operatorname{Ext}^{i}_{\Lambda_{\mathfrak{p}}}(M_{\mathfrak{p}}, -).$$

It follows from this that $\operatorname{projdim}_{\Lambda_\mathfrak{p}} M_\mathfrak{p}/x_1 M_\mathfrak{p} = \operatorname{projdim}_{\Lambda_\mathfrak{p}} M_\mathfrak{p} + 1$, and continuing we get

$$\operatorname{projdim}_{\Lambda_\mathfrak{p}} M_\mathfrak{p}/(x_1,\dots,x_t)M_\mathfrak{p} = \operatorname{projdim}_{\Lambda_\mathfrak{p}} M_\mathfrak{p} + \dim(R_\mathfrak{p}).$$

Since $\operatorname{gldim}\Lambda_\mathfrak{p} \leqslant n + \dim(R_\mathfrak{p})$, it must be that

$$\operatorname{projdim}_{\Lambda_\mathfrak{p}} M_\mathfrak{p} \leqslant n.$$

□

From this we get the following useful characterization of n-isolated singularities.

Corollary 4.4. *Let Λ be an order over a CM local ring R. Then Λ is an n-isolated singularity if and only if for all $X \in \mathcal{S}$, $X_\mathfrak{p}$ is a projective $\Lambda_\mathfrak{p}$-module for all non-maximal primes $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. ($\Rightarrow$): This follows at once from the previous lemma.

($\Leftarrow$): Fix $\mathfrak{p} \in \operatorname{Spec} R$. We begin by showing $\operatorname{gldim}\Lambda_\mathfrak{p} < \infty$. Let $M \in \operatorname{mod}\Lambda_\mathfrak{p}$, hence $M \in \operatorname{mod}\Lambda$. Let

$$\dots \longrightarrow P_k \longrightarrow P_{k-1} \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be a projective resolution of M. Then since $\Omega^d M$ is maximal Cohen-Macaulay (over R), we know $\Omega^{n+d} M \in \mathcal{S}$. Then it follows that $(\Omega^{n+d}M)_\mathfrak{p} = \Omega^{n+d} M_\mathfrak{p} \cong \Omega^{n+d} M$ is a projective module over $\Lambda_\mathfrak{p}$ by assumption. Thus $\operatorname{gldim}\Lambda_\mathfrak{p} < \infty$. Now, by Theorem 3.2, it suffices to show $\operatorname{projdim}\omega_{\Lambda_\mathfrak{p}} \leqslant n$. Since $\omega_{\Lambda_\mathfrak{p}} \cong (\omega_\Lambda)_\mathfrak{p}$ and $\omega_\Lambda \in \operatorname{CM}\Lambda$, it is clear that $\Omega^n(\omega_{\Lambda_\mathfrak{p}}) = (\Omega^n \omega_\Lambda)_\mathfrak{p}$. Hence $\Omega^n(\omega_{\Lambda_\mathfrak{p}})$ is projective by assumption. Thus, $\operatorname{projdim}_{\Lambda_\mathfrak{p}} \omega_{\Lambda_\mathfrak{p}} \leqslant n$. □

The following lemma will be useful later, as it detects n-isolated singularities.

Lemma 4.5. *Let R be a CM local ring with canonical module ω. Let Λ be an R-order. Then Λ is an n-isolated singularity if and only if $\ell_R(\operatorname{Ext}^1_\Lambda(N,M)) < \infty$ for all $M,N \in \mathcal{S}$.*

Proof. The necessity follows at once from Lemma 4.3. We prove the sufficiency. Suppose $\ell(\operatorname{Ext}^1_\Lambda(N,M)) < \infty$ for all $M,N \in \mathcal{S}$. Let $\mathfrak{p}$ be a prime ideal of R which is not maximal. Consider a module $M \in \operatorname{CM}\Lambda_\mathfrak{p}$. We wish to show $\operatorname{projdim}_{\Lambda_\mathfrak{q}} M \leqslant n$. Let X be the n^{th} syzygy of M over Λ. Since $R_\mathfrak{p}$ is R-flat and $\Lambda_\mathfrak{p} \cong \Lambda \otimes_R R_\mathfrak{p}$ as R-orders, we have $X_\mathfrak{p} = (\Omega^n_\Lambda M)_\mathfrak{p} = \Omega^n_{\Lambda_\mathfrak{p}} M_\mathfrak{p} = \Omega^n_{\Lambda_\mathfrak{p}} M$. We must only show, then, that $X_\mathfrak{p}$ is $\Lambda_\mathfrak{p}$-projective. Consider the exact sequence over $\Lambda_\mathfrak{p}$,

$$0 \longrightarrow \Omega(X_\mathfrak{p}) \longrightarrow F \longrightarrow X_\mathfrak{p} \longrightarrow 0, \tag{4.1}$$

where F is a free $\Lambda_\mathfrak{p}$-module. Since $\Omega(X_\mathfrak{p}) = (\Omega X)_\mathfrak{p}$ and $X, \Omega X \in \mathcal{S}$, it follows that

$$\operatorname{Ext}^1_{\Lambda_\mathfrak{p}}(X_\mathfrak{p}, \Omega X_\mathfrak{p}) \cong \operatorname{Ext}^1_\Lambda(X, \Omega X)_\mathfrak{p} = 0.$$

Where the final equality follows since $\operatorname{Ext}^1_\Lambda(X,\Omega X)$ has finite length by assumption. This means sequence 4.1 splits, and hence $X_\mathfrak{p}$ is $\Lambda_\mathfrak{p}$-projective, as desired.

□

The next proposition illustrates that n^{th} syzygies (of MCM modules) over an n-isolated singularity behave like MCM modules over an isolated singularity. This is shown for the $n = 0$ case in [8, Theorem 1.3.1], the proof is largely the same except the $d = 2$ case of part (1).

Proposition 4.6. *Let Λ be an n-isolated singularity over a d-dimensional CM local ring R. For $X \in \mathcal{S}$:*

(1) $\operatorname{Ext}^i_\Lambda(\operatorname{Tr} X^{\mathrm{op}}, \Lambda) = 0$ *for* $i = 1, \ldots, d$.
(2) $\operatorname{Ext}^i_\Lambda(X, Y)$, $\operatorname{Tor}^\Lambda_i(Z, X)$, *and* $\underline{\operatorname{Hom}}_\Lambda(X, Y)$ *are all finite length for any* $Y \in \operatorname{mod} \Lambda$ *and* $Z \in \operatorname{mod} \Lambda^{\mathrm{op}}$.

Proof. Let $\mathfrak{p} \in \operatorname{Spec} R$ be non-maximal. We see if $X \in \mathcal{S}$, then $X_\mathfrak{p}$ is projective over Λ_p by Lemma 4.3, and thus (2) holds. For assertion (1), we note that if $d = 0$ there is nothing to show. In the case where $d = 1$, the fact that X is projective on the punctured spectrum implies that $\operatorname{Ext}^1_\Lambda(\operatorname{Tr} X^{\mathrm{op}}, \Lambda)$ has finite length since $\operatorname{Tr} X^{\mathrm{op}}_\mathfrak{p} = 0$ for any non-maximal prime ideal $\mathfrak{p}$. Then the well-known exact sequence (see, e.g., [11, Proposition 12.8])

$$0 \longrightarrow \operatorname{Ext}^1_\Lambda(\operatorname{Tr} X^{\mathrm{op}}, \Lambda) \longrightarrow X \longrightarrow X^{**} \longrightarrow \operatorname{Ext}^2_\Lambda(\operatorname{Tr} X^{\mathrm{op}}, \Lambda) \longrightarrow 0$$

shows that $\operatorname{Ext}^1_\Lambda(\operatorname{Tr} X^{\mathrm{op}}, \Lambda)$ embeds in X. But, $\operatorname{depth}_R X \geqslant 1$ since $d \geqslant 1$ so X cannot contain a module of depth zero. Thus, $\operatorname{Ext}^1_\Lambda(\operatorname{Tr} X^{\mathrm{op}}, \Lambda) = 0$.

Now suppose $d \geqslant 2$. We still have that $\operatorname{Ext}^1_\Lambda(\operatorname{Tr} X^{\mathrm{op}}, \Lambda) = 0$ by the above case. Thus, we have an exact sequence

$$0 \longrightarrow X \longrightarrow X^{**} \longrightarrow \operatorname{Ext}^2_\Lambda(\operatorname{Tr} X^{\mathrm{op}}, \Lambda) \longrightarrow 0.$$

By virtue of being a dual module, X^{**} is a second syzygy and hence has depth over R at least 2. Since $\operatorname{Ext}^2_\Lambda(\operatorname{Tr} X^{\mathrm{op}}, \Lambda)$ is of finite length, hence depth zero over R, the Depth Lemma implies $\operatorname{depth}_R X = 1$. This is a contradiction since $d \geqslant 2$ and $X \in \operatorname{CM} \Lambda$. Thus, we must have $\operatorname{Ext}^2_\Lambda(\operatorname{Tr} X^{\mathrm{op}}, \Lambda) = 0$. It now follows from the above exact sequence that $X \cong X^{**}$.

Finally, suppose $\operatorname{Ext}^i_\Lambda(\operatorname{Tr} X^{op}, \Lambda) = 0$ for $i = 1, \ldots, k-1$ for some $3 \leqslant k \leqslant d$. We begin with a projective resolution

$$\ldots \longrightarrow P_k \longrightarrow P_{k-1} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \operatorname{Tr} X^{\mathrm{op}} \longrightarrow 0.$$

Dualizing the above exact sequence, and utilizing the fact that $X \cong X^{**}$, we get an exact sequence

$$0 \longrightarrow X \longrightarrow P_2^* \longrightarrow P_3^* \longrightarrow \ldots \longrightarrow P_{k-1}^* \longrightarrow (\Omega^k X)^* \longrightarrow \operatorname{Ext}^k_\Lambda(\operatorname{Tr} X^{\mathrm{op}}, \Lambda) \longrightarrow 0,$$

where $\operatorname{depth}_R(\Omega^k X)^* \geqslant 2$. Since $\operatorname{Ext}^k_\Lambda(\operatorname{Tr} X^{\mathrm{op}}, \Lambda)$ has finite length and $P_i^* \in \operatorname{CM} \Lambda$ for all i, the Depth Lemma implies $\operatorname{depth}_R X \leqslant d - 1$, which is impossible since $X \in \operatorname{CM} \Lambda$. Thus, it must be that $\operatorname{Ext}^k_\Lambda(\operatorname{Tr} X^{\mathrm{op}}, \Lambda) = 0$. Thus part (1) is proved by induction. $\square$

The following is the analog of [9, Prop 2.17], and the proof is largely the same.

Proposition 4.7. *Let Λ be an order over a CM ring R of Krull dimension d with canonical module ω_R. The following are equivalent:*

(1) Λ *is n-nonsingular.*

(2) $\operatorname{gldim}\Lambda_{\mathfrak{m}} \leqslant n+d$ *for all maximal ideals* $\mathfrak{m} \in \operatorname{Spec} R$.

(3) $\operatorname{CM}\Lambda \subset \operatorname{projdim}_{\leqslant n}\Lambda$.

(4) $\operatorname{projdim}_{\Lambda^{\mathrm{op}}}\omega_\Lambda \leqslant n$ *and* $\operatorname{gldim}\Lambda < \infty$.

Proof. The first 3 implications are the same argument as [9], but we include them for the convenience of the reader. (1) $\Rightarrow$ (2) This is immediate.

(2) $\Rightarrow$ (3) This proof is nearly identical to the proof of Lemma 4.3.

(3) $\Rightarrow$ (4) Since $\omega_\Lambda \in \operatorname{CM}\Lambda$, we know it has projective dimension at most n by (3). Further, since each d^{th} syzygy is MCM by the Depth Lemma, we also have $\operatorname{gldim}\Lambda < \infty$.

(4) $\Rightarrow$ (1) Let X be in $\operatorname{CM}(\Lambda_{\mathfrak{p}})$. We will show that $\operatorname{projdim}_{\Lambda_{\mathfrak{p}}} X \leqslant n$ and the Depth Lemma will conclude the proof as in the previous step. Since localization can only reduce projective dimension, we have that $\operatorname{projdim}_{\Lambda_{\mathfrak{p}}^{\mathrm{op}}}\omega_{\Lambda_{\mathfrak{p}}} \leqslant n$ and $\operatorname{gldim}\Lambda_{\mathfrak{p}} < \infty$. The result then follows from Theorem 3.2 □

Remark 4.8. *One might ask if we can strengthen condition (3) to be a set equality. If $n \geqslant 1$, the answer for (3) is no: consider a regular sequence $\underline{x} = x_1, \ldots, x_d$ on Λ, and take the Koszul complex over Λ on $\underline{x}$. Then this is exact, and has length d. Then $\Omega^{d-1}(\Lambda/\underline{x}\Lambda)$ has depth $d-1$ by the Depth Lemma, but the end of the Koszul complex gives a length one resolution. Thus $\Omega^{d-1}(\Lambda/\underline{x}\Lambda) \in \operatorname{projdim}_{\leqslant n}\Lambda$ but is not in $\operatorname{CM}\Lambda$. It is obvious that* (3) *is equivalent to $\mathcal{S} \subset \operatorname{Proj}\Lambda$. It is not clear when $\mathcal{S} = \operatorname{Proj}\Lambda$.*

5. Gorenstein Projectives and Auslander's Theorem. The goal of this section is to show the following variation of Auslander's Theorem, [1].

Theorem 5.1. *Let R be a CM local ring with canonical module and suppose Λ is n-canonical R-order. If Λ has only finitely many nonisomorphic indecomposable modules in $\mathcal{S}$, then Λ is an n-isolated singularity.*

The proof of this will rely on the notion of *Gorenstein Projective* modules. Originally defined by Auslander and Bridger in [2], a module M over an order Λ is called *Gorenstein Projective* if M is reflexive (i.e., the natural map $M \longrightarrow M^{**}$ is an isomorphism) and

$$\operatorname{Ext}^i_\Lambda(M,\Lambda) = \operatorname{Ext}^i_\Lambda(M^*,\Lambda) = 0$$

for all $i > 0$.

We let $G\operatorname{Proj}\Lambda$ denote the full subcategory of $\operatorname{mod}\Lambda$ consisting of all Gorenstein projective modules, and $\underline{G\operatorname{Proj}}\Lambda$ the corresponding stable category. Our interest in Gorenstein projectives is motivated by the following fact.

Proposition 5.2. *Let R be a CM local ring with canonical module ω. Suppose Λ is an n-canonical R-order, where $n \geqslant 2$. Let M be a non-projective Λ-module; then $M \in G\operatorname{Proj}$ if and only if $M \in \mathcal{S}$.*

Before the proof we note that the only reason we require M to be non-projective is that it is not necessarily true that $\operatorname{Proj}\Lambda \subset \mathcal{S}$ (see Remark 4.8), but certainly all projectives are also Gorenstein Projective.

Proof. Since Gorenstein projectives occur as syzygies in complete resolutions, it is clear that

$$G\operatorname{Proj}\Lambda \setminus \operatorname{add}\Lambda \subset \operatorname{add}\Omega^n(\operatorname{CM}\Lambda).$$

We show the reverse inclusion. Let $M = \Omega^n X$ for a maximal Cohen-Macaulay module X, and suppose M is not a projective module. By Lemma 3.6 we have that $\operatorname{Ext}^i_\Lambda(M,\Lambda) = 0$ for all $i > 0$. Then, by dualizing a projective resolution of M, we get an exact sequence

$$0 \longrightarrow M^* \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \cdots.$$

According to Lemma 4.3, M necessarily has infinite projective dimension if it is not projective; therefore, we see M^* is an arbitrarily high syzygy. By Lemma 3.6 again we have $\operatorname{Ext}^i_\Lambda(M^*,\Lambda) = 0$ for $i > 0$. All that remains to show is that M is reflexive. Note that $\operatorname{Tr}M^{\mathrm{op}}$ fits into the above exact sequence as follows

$$0 \longrightarrow \operatorname{Tr}M^{\mathrm{op}} \longrightarrow P_2^* \longrightarrow P_3^* \longrightarrow \cdots.$$

Thus, $\operatorname{Tr}M^{\mathrm{op}}$ is also an arbitrarily high syzygy and satisfies the same Ext vanishing as M. Thus the exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_\Lambda(\operatorname{Tr}M^{\mathrm{op}},\Lambda) \longrightarrow M \longrightarrow M^{**} \longrightarrow \operatorname{Ext}^2_\Lambda(\operatorname{Tr}M^{\mathrm{op}},\Lambda) \longrightarrow 0$$

implies that $M \cong M^{**}$.

□

The key use of Gorenstein projectives is that they are closed under extensions. This has been shown in various places, see e.g., [4, Proposition 5.1].

Corollary 5.3. *Let Λ be a n-canonical order over a CM local ring R with canonical module ω. Then $\mathcal{S}$ is closed under extensions.*

We now return to proving the main theorem. The proof of this involves several lemmas. It follows closely Huneke and Leuschke's proof of Auslander's Theorem, [7]. The following Theorem due to Miyata is our first step.

Lemma 5.4. [13, Theorem 2] *Let A be a module finite algebra over a commutative Noetherian ring R. Suppose we have an exact sequence of finitely generated Λ-modules*

$$M \longrightarrow X \longrightarrow N \longrightarrow 0$$

and that $X \cong M \oplus N$. Then the sequence is a split short exact sequence.

From this we are able to deduce the following lemma about $\mathrm{Ext}^1_\Lambda(N,M)$; the proof is similar to the one in [7]; it is ommitted for this reason.

Lemma 5.5. *Let $(R,\mathfrak{m})$ be a CM local ring and Λ an R-order. Fix $r \in \mathfrak{m}$. Suppose we have an exact sequence of Λ-modules,*

$$\alpha:\ 0 \longrightarrow M \longrightarrow X_\alpha \longrightarrow N \longrightarrow 0$$

and a commutative diagram

$$\begin{tikzcd}
\alpha:\ 0 \arrow[r] & M \arrow[r] \arrow[d, "r"'] & X_\alpha \arrow[r] \arrow[d, "f"'] & N \arrow[r] \arrow[d, equal] & 0 \\
r\alpha:\ 0 \arrow[r] & M \arrow[r] & X_{r\alpha} \arrow[r] & N \arrow[r] & 0.
\end{tikzcd}$$

If $X_\alpha \cong X_{r\alpha}$, then $\alpha \in r\,\mathrm{Ext}^1_\Lambda(N,M)$.

Now, we are able to prove the following lemma from which the main theorem follows. The proof is a straightforward generalization of the commutative case but is included for convenience.

Lemma 5.6. *Suppose Λ is an order over a CM local ring $(R,\mathfrak{m},k)$. Given Λ-modules M and N, if there are only finitely many choices (up to isomorphism) for X such that there is an exact sequence of Λ-modules*

$$0 \longrightarrow M \longrightarrow X \longrightarrow N \longrightarrow 0,$$

then $\mathrm{Ext}^i_\Lambda(N,M)$ is a finite length R-module.

Proof. Let $\alpha \in \mathrm{Ext}^1_\Lambda(N,M)$ and $r \in \mathfrak{m}$. It is well known that an R-module M has finite length if and only if for all $r \in \mathfrak{m}$ and $x \in M$ there is an integer n so that $r^n x = 0$. Thus, we must only show that $r^n\alpha = 0$ for $n \gg 0$. For any integer n we consider a representative

$$r^n\alpha:\quad 0 \longrightarrow M \longrightarrow X_n \longrightarrow N \longrightarrow 0.$$

Since only finitely many X_n can exist up to isomorphism there is an infinite sequence $n_1 < n_2 < n_3 < \dots$ such that $X_{n_i} \cong X_{n_j}$ for all pairs i,j. Set $\beta = r^{n_1}\alpha$, and let $i > 1$. Then $r^{n_i}\beta = r^{n_i - n_1}\alpha$. We show $\beta = 0$. We have, for each i, a commutative diagram

$$\begin{array}{ccccccccc} \beta:0 & \longrightarrow & M & \longrightarrow & X_{n_1} & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle r^{n_i-n_1}} & & \downarrow & & \| & & | \\ r^{n_i-n_1}\beta:0 & \longrightarrow & M & \longrightarrow & X_{n_i} & \longrightarrow & N & \longrightarrow & 0. \end{array}$$

By Lemma 5.5, since $X_{n_1} \cong X_{n_i}$, we have $\beta \in r^{n_i-n_1}\operatorname{Ext}^1_\Lambda(N,M)$ for every i. Since the sequence of n_i is infinite and strictly increasing, this means $\beta \in \mathfrak{m}^t \operatorname{Ext}^1_\Lambda(N,M)$ for all t. Finally, the Krull Intersection Theorem [12, Theorem 8.10] implies $\beta = 0$.

□

Finally, we provide the proof of the main theorem, Theorem 5.1.

Proof of Theorem 5.1. Let $M,N \in \mathcal{S}$. By Lemma 4.5 we must only show that $\ell_R(\operatorname{Ext}^1_\Lambda(N,M)) < \infty$. Consider any sequence $\alpha \in \operatorname{Ext}^1_\Lambda(N,M)$,

$$\alpha: 0 \longrightarrow M \longrightarrow X \longrightarrow N \longrightarrow 0.$$

By Corollary 5.3, we know $X \in \mathcal{S}$. Now since M and N are finitely generated and there are only finitely many indecomposable modules in $\mathcal{S}$, there are only finitely many possibilities for X. Namely, X must be one of the finitely many modules in $\mathcal{S}$ generated by at most $\mu_\Lambda(M)+\mu_\Lambda(N)$ where $\mu_\Lambda(Y)$ denotes the minimum number of generators of Y over Λ. Thus, $\ell(\operatorname{Ext}^1_\Lambda(N,M)) < \infty$ by Lemma 5.6.

□

6. Application to Commutative Rings. In view of Theorem 5.1 and 3.16, we arrive at the following generalization of Auslander's Theorem in the case where R is a suitable Gorenstein local ring.

Corollary 6.1. *Let R be a Gorenstein local ring which is an order over a regular local ring S (e.g., if R is complete), and let Q an acyclic quiver. If there exist only finitely many nonisomorphic indecomposable modules in $\Omega\operatorname{CM}(RQ)$, then R is an isolated singularity, i.e.,*

$$\operatorname{gldim} R_\mathfrak{p} = \dim(R_p)$$

for all non-maximal primes ideals $\mathfrak{p} \in \operatorname{Spec} R$.

Proof. We only need to notice that by Theorem 3.16 RQ is a 1-canonical order. Thus by Theorem 5.1 if there are only finitely many indecomposable modules in $\Omega^n\operatorname{CM}(RQ)$ we must have that RQ is a 1-isolated singularity. It is well known (see e.g., [14]) that $\operatorname{gldim} RQ < \infty$ if and only if $\operatorname{gldim} R < \infty$ for any commutative ring R. Thus, RQ can be a 1-isolated singularity if and only if $\operatorname{gldim} R_\mathfrak{p} < \infty$ for all non-maximal primes $\mathfrak{p}$. Since R is commutative, this is only possible if $\operatorname{gldim} R_\mathfrak{p} = \dim(R_\mathfrak{p})$.

□

7. Further Questions. There are many questions relating to n-canonical orders and n-isolated singularities. Here we include two for consideration.

Our example of an n-canonical order of infinite global dimension is not an isolated singularity. Orders which are isolated singularities are particularly interesting, especially in regards to noncommutative desingularizations. Thus, we arrive at the following questions.

Question 7.1. *Is it possible for an n-canonical order of infinite global dimension to be an isolated singularity? Is it possible to have an n-canonical order of infinite global dimension over a Cohen-Macaulay ring (with canonical module) which is not Gorenstein?*

We note that the proof of Corollary 6.1 does not require completeness beyond ensuring R is an order over a regular local ring. It would be nice to remove this assumption. In this vein we have the following question:

Question 7.2. *For a local ring R and an acyclic quiver Q, is it true that RQ has only finitely many indecomposable modules in $\mathcal{S}$ if and only if $\widehat{RQ} \cong \widehat{R}Q$ has?*

This question does not appear to be a straightforward generalization of the techniques used by Wiegand in [16]. Even for path algebras, ascent from RQ to the henzelisation R^hQ seems to be difficult.

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